Turbulent Flows
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Solution to Exercise 13.1

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Combining Eq.(13.1) with the temporal derivative of the filtered velocity field $\overline{\mathbf{U}}(\mathbf{x},t)$ leads

$$\begin{aligned}\frac{\partial \overline{\mathbf{U}}}{\partial t} &= \frac{\partial}{\partial t}\left(\int G(\mathbf{r},\mathbf{x})\mathbf{U}(\mathbf{x}-\mathbf{r},t)\,d\mathbf{r}\right)\\ &= \int G(\mathbf{r},\mathbf{x})\frac{\partial \mathbf{U}(\mathbf{x}-\mathbf{r},t)}{\partial t}\,d\mathbf{r}\\ &= \overline{\left(\frac{\partial \mathbf{U}}{\partial t}\right)},\end{aligned} \tag{1}$$

which is equivalent to Eq.(13.6).

Starting from $\langle\overline{\mathbf{U}}\rangle$ combined with Eq.(13.1) one has

$$\begin{aligned}\langle\overline{\mathbf{U}}\rangle &= \left\langle\int G(\mathbf{r},\mathbf{x})\mathbf{U}(\mathbf{x}-\mathbf{r},t)\,d\mathbf{r}\right\rangle\\ &= \left\langle\lim_{n\to\infty}\sum_{\mathbf{k}} G(\mathbf{r_k},\mathbf{x})\mathbf{U}(\mathbf{x}-\mathbf{r_k},t)\Delta\mathbf{r}\right\rangle\\ &= \lim_{n\to\infty}\sum_{\mathbf{k}} G(\mathbf{r_k},\mathbf{x})\langle\mathbf{U}(\mathbf{x}-\mathbf{r_k},t)\rangle\Delta\mathbf{r}\\ &= \int G(\mathbf{r},\mathbf{x})\langle\mathbf{U}(\mathbf{x}-d\mathbf{r},t)\rangle\,d\mathbf{r}\\ &= \overline{(\langle\mathbf{U}\rangle)}.\end{aligned} \tag{2}$$

In the second step, the triple integral over the displacement vector $d\mathbf{r}$ was replaced by a triple sum similarly to

$$\int_a^b f(r)\,dr = \lim_{n\to\infty}\sum_{k=1}^{n} f(r_k)\Delta r \tag{3}$$

for one dimension, an arbitrary function $f(r)$, $r_k = a+k\Delta r$ and $\Delta r = \frac{b-a}{n}$. In the third step, the linearity properties Eq.(3.26) and Eq.(3.27) of the average could then be used, resulting in the relation given by Eq.(13.7).

Similarly to Eq.(1), the spatial derivative can be written with Eq.(13.1)

$$\begin{aligned}
\frac{\partial \overline{U_i}}{\partial x_j} &= \frac{\partial}{\partial x_j}\left(\int G(\mathbf{r},\mathbf{x})U_i(\mathbf{x}-\mathbf{r},t)\,d\mathbf{r}\right) \\
&= \int \frac{\partial}{\partial x_j}\left(G(\mathbf{r},\mathbf{x})U_i(\mathbf{x}-\mathbf{r},t)\right)\,d\mathbf{r} \\
&= \int G(\mathbf{r},\mathbf{x})\frac{\partial U_i(\mathbf{x}-\mathbf{r},t)}{\partial x_j}\,d\mathbf{r} + \int \frac{\partial G(\mathbf{r},\mathbf{x})}{\partial x_j}U_i(\mathbf{x}-\mathbf{r},t)\,d\mathbf{r} \\
&= \overline{\left(\frac{\partial U_i}{\partial x_j}\right)} + \int \frac{\partial G(\mathbf{r},\mathbf{x})}{\partial x_j}U_i(\mathbf{x}-\mathbf{r},t)\,d\mathbf{r}, \qquad (4)
\end{aligned}$$

which is Eq.(13.8).